Linear Programming - 4th part

Initial Basis – Duality – Applications

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Outline

Find an initial solution

- Let an orange juice manufacture, that sells a drink made of soda and orange juice ;
- Each dl of soda contains 0.05kg of sugar and 1g of vitamin C ;
- Each dl of orange juice contains 0.025kg of sugar and 3g of vitamin C ;
- Each dl of soda costs 2 centimes and each dl of orange juice costs 3 centimes ;
- Each bottle of final drink has a volume of 1l and must contain at least 20g of vitamin C and at most 0.4kg of sugar.
- Find a way to produce this drink at a minimal cost.

Standard form

- Let x_1= nb of dl of soda, x_2= nb of dl of O.J.
- $$\begin{array}{rrrrrl} \text{minimize } z = & 2\,x_1 & + & 3\,x_2 & & & \\ & 0.5\,x_1 & + & 0.25\,x_2 & + x_3 & & = 4 \\ & x_1 & + & 3\,x_2 & & - x_4 & = 20 \\ & x_1 & + & x_2 & & & = 10 \end{array}$$

  All the variables are positive.
- How can we find an initial feasible basis system (FBS) ?

Adding variables

- It is always possible to add other artificial variables, for instance:

$$
\begin{array}{lrcrcrcrcrcrl}
\text{minimize } z = & 2\,x_1 & + & 3\,x_2 & & & & & & & & \\
 & 0.5\,x_1 & + & 0.25\,x_2 & + & x_3 & & & & & & = 4 \\
 & x_1 & + & 3\,x_2 & & & - & x_4 & + & x_5 & & = 20 \\
 & x_1 & + & x_2 & & & & & & & + \; x_6 & = 10
\end{array}
$$

- **NOTE** It is not sufficient to add a single variable on the last line. Indeed, in this case, x_4 would be negative and so could not be part of an initial FBS.
- This is useful, but the system may well converge towards a solution which is not part of the initial problem, for instance x_3 =4, x_4=20, x_5=10, $x_1 = x_2 = 0$.

Cost of artificial variables

- To avoid this problem, we may add a heavy cost to the extra variables, so that they are not part of the optimal solution (i.e. their value will converge to zero), for instance:

$$z = 2\,x_1 + 3\,x_2 + M\,x_5 + M\,x_6$$

With M positive and sufficiently large.

Iteration 1 – variable addition

Here we consider $M = 100$.

- IBV=$\{x_3, x_5, x_6\}$; NBV=$\{x_1, x_2, x_4\}$
- $\bar{b} = \begin{bmatrix} 4 & 20 & 10 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} -198 & -397 & 100 \end{bmatrix}$ So x_2 enters the basis
- $P = \begin{bmatrix} 0.25 & 3 & 1 \end{bmatrix}$
- Ratios = $\begin{bmatrix} 16 & 6 & 10 \end{bmatrix}$ so x_5 exits the basis.

Iteration 2 – variable addition

- IBV=$\{x_3, x_2, x_6\}$; NBV=$\{x_1, x_4, x_5\}$
- $\bar{b} = \left[\, 2.333 \;\; 6.667 \;\; 3.333 \,\right]$
- Reduced costs= $\left[\, -65.667 \;\; -32.333 \;\; 132.333 \,\right]$ so x_1 enters,
- $P = \left[\, 0.417 \;\; 0.333 \;\; 0.667 \,\right]$
- Ratios = $\left[\, 5.6 \;\; 20 \;\; 5 \,\right]$ so x_6 exits.

Iteration 3 – variable addition

- IBV=$\{x_3, x_2, x_1\}$; NBV=$\{x_4, x_5, x_6\}$
- $\bar{b} = \begin{bmatrix} 0.25 & 5 & 5 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} 0.5 & 99.5 & 98.5 \end{bmatrix}$
- Optimal solution =

$$\begin{bmatrix} x_3 = 0.25 \\ x_2 = 5 \\ x_1 = 5 \end{bmatrix}$$

$z = 25$.

Impossible solutions

- In the case where we add artificial variables, it is possible that they may end up in the optimal solution even with a very high M. This indicates an unfeasible solution.
- For instance, if we demand that the final product contain 36g of vitamin C. Since the maximum feasible is 30g (with pure orange juice), this is clearly infeasible.
- However, with the extra variables, we find a solution in two iterations:

$$\begin{bmatrix} x_3 = 1.5 \\ x_5 = 6 \\ x_2 = 10 \end{bmatrix}$$

$z = 630$.
Since x_5 is artificial, this means this solution is not valid.

Two-step solution

- A good choice for M is about $100\times$ greater than the largest coefficient in the objective function. This may introduce numerical errors and imprecision.
- For this reason a two-step solution is preferable:
- In the first step, as before, we add artificial solutions, so that we find a trivial initial basis to the modified LP.
- We also modify the objective function: we now look to minimize the sum of the artificial variables we have just added. In this way, we will converge to a solution where these extra variables are set to zero, as they are no longer in basis.

Second step

- There are now three possibilities :
  1. The optimal objective (the sum of extra variables) is non-zero. This means the initial problem is unfeasible.
  2. The optimal objective is zero, and no extra variables is in the final basis. In this case, this final basis is taken as the initial basis of the initial LP, without the extra variables.
  3. The optimal objective is zero, however at least one of the artificial variable is in the final basis. In this case we take the final basis as initial basis for the initial LP, but we have to keep the extra variables that are in this basis.

Justification

- If the LP is not feasible, the only way to obtain a feasible solution in the modified LP would be to have at least one extra variable that is strictly positive. In this case the optimal modified objective function cannot be zero.
- On the other hand, if the original LP has a feasible solution, so this solution is also feasible in the modified LP, and in this case all the extra variable are necessarily set to zero, and this is optimal for the modified LP.
- In real problems, case 3 is extremely rare.

Iteration 1 – first phase

On the OJ vs. Soda problem :

- IBV=$\{x_3, x_5, x_6\}$; NBV=$\{x_1, x_2, x_4\}$
- $\bar{b} = \begin{bmatrix} 4 & 20 & 10 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} -2 & -4 & 1 \end{bmatrix}$ x_2 enters.
- $P = \begin{bmatrix} 0.25 & 3 & 1 \end{bmatrix}$
- Ratios = $\begin{bmatrix} 16 & 6.667 & 10 \end{bmatrix}$ x_5 exits.

Iteration 2 – added variables

- IBV=$\{x_3, x_2, x_6\}$; NBV=$\{x_1, x_4, x_5\}$
- $\bar{b} = \begin{bmatrix} 2.333 & 6.667 & 3.333 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} -0.667 & -0.333 & 1.333 \end{bmatrix}$ x_1 enters.
- $P = \begin{bmatrix} 0.417 & 0.333 & 0.667 \end{bmatrix}$
- Ratios = $\begin{bmatrix} 5.6 & 20 & 5 \end{bmatrix}$ x_6 exits.

Itération 3 – added variables

- IBV=$\{x_3, x_2, x_1\}$; NBV=$\{x_4, x_5, x_6\}$
- $\bar{b} = \begin{bmatrix} 0.25 & 5 & 5 \end{bmatrix}$
- Reduced costs= $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$
- Optimal solution =

$$\begin{bmatrix} x_3 = 0.25 \\ x_2 = 5 \\ x_1 = 5 \end{bmatrix}$$

$z' = 0$.

Phase II

- The solution from phase I is a FBS for the initial system.
- So we use it as an initial basis for the initial problem.
- Here, the initial basis happens to be optimal for the initial problem. However this is never guaranteed of course.
- In the case were the initial problem is unfeasible (e.g. the 36g of vitamin C example), we can verify that the solution that is found is the following:

$$\begin{bmatrix} x_3 = 1.5 \\ x_5 = 6 \\ x_2 = 10 \end{bmatrix}$$

$z' = 6$

Note: the same as with the M method.

Non-necessarily positives (nnp) variables

- The ratio tests imposes to keep only positive ratios, however this does not work in the cases where some or all variables may be legitimely negative.
- As before, we can solve this problem by adding variables to the problem.
- For each variable x_i that is non necessarily positive (nnp), we substitute this variable by $x_i' - x_i''$ and we add the constraint $x_i' \geq 0$ and $x_i'' \geq 0$.
- Now we are back to a standard simplex.
- No FBS may have both $x_i' > 0$ and $x_i'' > 0$ (why ?)

Nnp variables – next

- We have three cases :
  1. $x'_i > 0$ and $x''_i = 0$. In this case $x_i = x'_i$.
  2. $x'_i = 0$ and $x''_i > 0$. In this case $x_i = -x''_i$.
  3. $x'_i = 0$ and $x''_i = 0$. In this case $x_i = 0$.

Boulangerie revisited

- A baker has 30kg of flour et 5 packets of baking powder
- A baking set requires 5kg of flour and one packet of baking powder
- Each set sells for 30 Euros
- The baker may buy or sell flour at 4 Euros/kg.
- How can we maximize profit ?

Formulation

- x_1 = number of sets ; x_2 = number of kg of extra flour.
- If $x_2 > 0$ then the baker will have bought some flour, if $x_2 < 0$ he will have sold some.
- The LP is:

$$\begin{aligned} \text{maximize}\, z = 30\,x_1 &- 4\,x_2 \\ 5\,x_1 &- \quad x_2 \leq 30 \\ x_1 & \qquad\quad \leq 5 \end{aligned}$$

$x_1 \geq 0$, x_2 nnp.

Re-formulation

- We replace x_2 by $x_2' - x_2''$.
- In standard form:

$$
\begin{array}{rrrrrrl}
\text{maximize}\, z = & -30\,x_1 & +\ 4\,x_2' & -\ 4\,x_2'' & & & \\
& 5\,x_1 & -\quad x_2 & +\quad x_2'' & +\ x_3 & & = 30 \\
& x_1 & & & & +\ x_4 & = 5
\end{array}
$$

with $x_1, x_2', x_2'', x_3, x_4 \geq 0$

Solution

- In 3 iterations, we find the optimal solution:

$$\begin{bmatrix} x_1 = 5 \\ x_2'' = 5 \end{bmatrix}$$

$z = -170$

- I.e. $x_2 = -5$.
- Interpretation: the baker must sell 5kg of flour, indeed he is limited by the 5 packets of raising powder, and at the end of the production, there remains $30 - 25 = 5$kg that it is optimal to resell.

Consistency of variables

- The variables x'_i et x''_i cannot be simultaneously strictly positive
- Indeed the column vectors associated to x'_i et x''_i are opposite. They are not independent and so cannot simultaneously form part of a basis.
- At most one of these vectors can be part of a basis.

Duality: Primal

- Let a *primal* LP in standard form:

$$
\begin{array}{rcccccccl}
\text{maximize}\, z = & c_1 x_1 & + & c_2 x_2 & + \ldots + & c_n x_n & & \\
 & a_{11} x_1 & + & a_{12} x_2 & + \ldots + & a_{1n} x_n & \leq & b_1 \\
 & a_{21} x_1 & + & a_{22} x_2 & + \ldots + & a_{2n} x_n & \leq & b_2 \\
 & \vdots & + & \vdots & + & \vdots & & \\
 & a_{m1} x_1 & + & a_{m2} x_2 & + \ldots + & a_{mn} x_n & \leq & b_m
\end{array}
$$

$\forall i, x_i \geq 0$

- The dual of this problem become a minimization problem.

Dual problem

- This is the *dual* in standard form:

$$
\begin{array}{lcccccccl}
\text{minimize}\, w = & b_1 y_1 & + & b_2 y_2 & + \ldots + & b_m y_m & & \\
& a_{11} y_1 & + & a_{21} y_2 & + \ldots + & a_{m1} y_m & \geq c_1 \\
& a_{12} y_1 & + & a_{22} y_2 & + \ldots + & a_{m2} y_m & \geq c_2 \\
& \vdots & + & \vdots & + & \vdots & \\
& a_{1n} y_1 & + & a_{2n} y_2 & + \ldots + & a_{mn} y_m & \geq c_n
\end{array}
$$

$\forall i, y_i \geq 0$

- This formulation is derived from the *Lagrangian* dual.

Example – vitamins

- A person must absorb enough vitamins V et W every day
- These vitamins are available in everyday food, e.g. milk and eggs.
- Vitamin needs are expressed in this table:

Vitamin	Quantity in milk	Quantity in eggs	daily requirements
V	2	4	40
W	3	2	50
Unit cost	3	2.5	

- Minimize the cost of the diet

Formulation

x_1 = quantity of milk purchased ; x_2 = quantity of eggs purchased

$$\begin{aligned} \text{minimize}\, z = 3\,x_1 &+ 2.5\,x_2 \\ 2\,x_1 &+ 4\,x_2 - x_3 = 40 \\ 3\,x_1 &+ 2\,x_2 - x_4 = 50 \end{aligned}$$

$x_i \geq 0$.

Solution

- From the initial basis $\{x_1, x_2\}$, we find:

$$\begin{bmatrix} x_1 = 15 \\ x_2 = 2.5 \end{bmatrix}$$

  $z = 51.25$.

- This is a formulation from the point of view of the buyer. Let us look at the dual problem:

Dual formulation

y_1 = price of a unit of vitamin V ; y_2 = price of a unit of vitamin W.

$$
\begin{aligned}
\text{maximize } w = 40\,y_1 + 50\,y_2 & \\
2\,y_1 + 3\,y_2 + y_3 \quad & = 3 \\
4\,y_1 + 2\,y_2 \quad + y_4 & = 2.5
\end{aligned}
$$

$y_i \geq 0$

Solution

- from the initial basis $\{y_3, y_4\}$, we find in 3 iterations

$$\begin{bmatrix} y_1 = 0.1875 \\ y_2 = 0.8750 \end{bmatrix}$$

  $w = 51.25$

- This solution provides a solution from the point of view of the seller, who tries to sell vitamins V et W at the best possible price.

Benefits of the dual problem

- Interpretation
- May be (much) easier to solve if the primal problem has many constraints.
- The primal has a B with rank m (nb of constraints), the dual is of rank n (nb of variables). One if often smaller than the other.
- It is possible to use the intermediate solution of the primal to obtain a solution of the dual, and vice versa. This way we can alternate solving the dual and the primal, which makes it easier to limit the possible value of z, which is always between the intermediate solutions of the primal and of the dual.
- Feasible LP have no gap between the solution of the dual and the primal.

Things we have not studied:

- Robustness of solution
- Primal-dual algorithms
- Polynomial algorithms: interior point methods
- Manual methods: pivots
- Approximation of more complex problems by LP
- And much much more (unfortunately)