Linear programming – part 3

Limit cases for the simplex algorithm

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Outline

Application of the simplex

- The algorithm is described for a *minimization* of z, to maximize, just minimize $-z$.
- Finding an initial basis is not always trivial
- The algorithm yields a basis and one extreme value of the variables. The rest must be found by resolving $B^{-1}b$
- There are several limit cases, which we must take into account

Unique optimal solution

- A cabinet maker produce desks, tables and chairs
- Each kind requires wood, and two types of work : woodworking and finishing

Resource	desk	tables	chairs
planks	8m	6m	1m
woodwork	4h	2h	1.5h
finishing	2h	1.5h	0.5h

- We have 48m of wood planks, 20h of woodworking and 8h of finishing
- A desk yields a profit of 60 euros, a table 30 euros and a chair 20 euros
- Demand for chairs and desks is unlimited but we think there is a market for 5 tables at the most
- Maximize profit.

Formulation

- Variables : x_1 = nb. desks, x_2 = tables, x_3= chairs
- Objective : $\max z = 60x_1 + 30x_2 + 20x_3$
- Constraints :

$$
\begin{array}{rcrcrcr}
8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
 & & x_2 & & & \leq & 5
\end{array}
$$

$x_1, x_2, x_3 \geq 0$

Standard form

All inequalities are $\leq$, so we have slack variables (and not excess)

$$
\begin{array}{lrrrrrrrl}
\text{min } z = & -60x_1 & - 30x_2 & - 20x_3 & & & & & \\
 & 8x_1 & + 6x_2 & + x_3 & + x_4 & & & & = 48 \\
 & 4x_1 & + 2x_2 & + 1.5x_3 & & + x_5 & & & = 20 \\
 & 2x_1 & + 1.5x_2 & + 0.5x_3 & & & + x_6 & & = 8 \\
 & & x_2 & & & & & + x_7 & = 5
\end{array}
$$

First iteration

- Initially :

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 48 \\ 20 \\ 8 \\ 5 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -60 & -30 & -20 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Initial basis is IBV=$\{x_4, x_5, x_6, x_7\}$ Therefore we have

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 48 \\ 20 \\ 8 \\ 5 \end{bmatrix}, E = \begin{bmatrix} 8 & 6.0 & 1.0 \\ 4 & 2.0 & 1.5 \\ 2 & 1.5 & 0.5 \\ 0 & 1.0 & 0.0 \end{bmatrix}$$

First iteration – change of variables

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

	x_1	x_2	x_3
$c_e^\intercal$ =	[-60	-30	-20]

  Therefore x_1 (the min) enters the basis.

- $P = B^{-1} A_1 = [\,8\;\;4\;\;2\;\;0\,]^\intercal$
- ratios :

	x_4	x_5	x_6	x_7
$\frac{\bar{b}}{P} =$	6	5	4	∞

  therefore x_6 (the positive min) exits the basis.

Second iteration

- We have now IBV=$\{x_4, x_5, x_1, x_7\}$

$$B = \begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 16 \\ 4 \\ 4 \\ 5 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 0 \\ 2 & 1.5 & 0 \\ 1.5 & 0.5 & 1 \\ 1 & 0.0 & 0 \end{bmatrix}$$

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

$$c_e^\intercal = \begin{matrix} x_2 & x_3 & x_6 \\ [\ 15 & -5 & 30\ ] \end{matrix}$$

Therefore x_3 (the min) enters the basis

- $P = B^{-1} A_3 = [\,-1\ 0.5\ 0.25\ 0\,]^\intercal$
- ratios :

$$\frac{\bar{b}}{P} = \begin{matrix} x_4 & x_5 & x_1 & x_7 \\ -16 & 8 & 16 & \infty \end{matrix}$$

, therefore x_5 (the min for which P is positive) exists the basis

Third iteration

- We have now IBV=$\{x_4, x_3, x_1, x_7\}$

$$B = \begin{bmatrix} 1 & 1.0 & 8 & 0 \\ 0 & 1.5 & 4 & 0 \\ 0 & 0.5 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 24 \\ 8 \\ 2 \\ 5 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

$$\begin{array}{cccc} & x_2 & x_5 & x_6 \\ c_e^\intercal = & [\,5 & 10 & 10\,] \end{array}$$

This is the optimum !

Solution

- The solution is made up of variables in the basis solution, here

  IBV=$\{x_4, x_3, x_1, x_7\}$

- The values of these variables are given by

$$\bar{b} = \{24, 8, 2, 5\}$$

  respectively. All the others are at zero
- The cost function is therefore :

$$z = -60x_1 - 30x_2 - 20x_3 = -280$$

Multiple optimal solutions

- We consider the same problem, only this time we assume a table returns a profit of 35 euros instead of 30
- The rest is unchanged

First iteration (table=35 Euros)

- Initally IBV=$\{x_4, x_5, x_6, x_7\}$ which yields

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 48 \\ 20 \\ 8 \\ 5 \end{bmatrix}, E = \begin{bmatrix} 8 & 6 & 1 \\ 4 & 6 & 1.5 \\ 2 & 1.5 & 0.5 \\ 0 & 1.0 & 0 \end{bmatrix}$$

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

$$c_e^\intercal = \begin{array}{ccc} x_1 & x_2 & x_3 \\ [\ -60 & -35 & -20\ ] \end{array}$$

  Therefore x_1 (the min) enters the basis

- $P = B^{-1} A_1 = [\,8\ \ 4\ \ 4\ \ 0\,]^\intercal$
- ratios :

$$\frac{\bar{b}}{P} = \begin{array}{cccc} x_4 & x_5 & x_6 & x_7 \\ 6 & 5 & 4 & \infty \end{array}$$

  therefore x_6 (the min so that P is positive) exists the basis

Second iteration (table at 35 Euros)

- We have now IBV=$\{x_4, x_5, x_1, x_7\}$ which yields

$$B = \begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 16 \\ 4 \\ 4 \\ 5 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 0 \\ 2 & 1.5 & 0 \\ 1.5 & 0.5 & 1 \\ 1 & 0.0 & 0 \end{bmatrix}$$

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

	x_2	x_3	x_6
$c_e^\intercal$ =	[10	-5	30]

Therefore x_3 (the min) enters the bass.

- $P = B^{-1} A_3 = [\,-1\ 0.5\ 0.25\ 0\,]^\intercal$
- ratios :

	x_4	x_5	x_1	x_7
$\frac{\bar{b}}{P} =$	-16	8	16	∞

so x_5 (the min such that P is positive) exits the basis

Third iteration (tables at 35 Euros)

- We have now IBV=$\{x_4, x_3, x_1, x_7\}$ therefore

$$B = \begin{bmatrix} 1 & 1.0 & 8 & 0 \\ 0 & 1.5 & 4 & 0 \\ 0 & 0.5 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 24 \\ 8 \\ 2 \\ 5 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

$$c_e^\intercal = \begin{array}{ccc} x_2 & x_5 & x_6 \\ [\,0 & 10 & 10\,] \end{array}$$

We have found *an* optimum, but x_2 is zero, this indicates a non-unique solution. We can now make x_2 enter at a constant cost.
Therefore x_2 (the min) enters the basis.

Third iteration, continued

- We continue the calculation normally :
- $P = B^{-1}A_3 = [\,-2\ \ -2\ \ 1.25\ \ 1\,]^\mathsf{T}$
- ratios :

	x_4	x_3	x_1	x_7
$\frac{\bar{b}}{P} =$	-12	-4	1.6	5

So x_1 (the min for which P is positive) exits the basis

Fourth iteration (tables at 35 Euros)

- We now have IBV=$\{x_4, x_3, x_2, x_7\}$ As a result

$$B = \begin{bmatrix} 1 & 1.0 & 6 & 0 \\ 0 & 1.5 & 2 & 0 \\ 0 & 0.5 & 1.5 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \bar{b} = \begin{bmatrix} 27.2 \\ 11.2 \\ 1.6 \\ 3.4 \end{bmatrix}, E = \begin{bmatrix} 8 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Reduced costs $\bar{c}_e^\intercal = c_e^\intercal - c_b^\intercal B^{-1} E$

$$\begin{matrix} & x_1 & x_5 & x_6 \\ c_e^\intercal = & [\,0 & 10 & 10\,] \end{matrix}$$

We have found an *un* optimum, but x_1 is zero. This indicates a non-unique solution, since x_1 can enter the basis with a constant cost.

Fourth iteration – the end

If x_1 enters the basis again, we have the same solution as above :

- $P = B^{-1}A_3 = [\,-2\ -2\ 1.25\ 1\,]^\mathsf{T}$
- ratios :

	x_4	x_3	x_2	x_7
$\frac{\bar{b}}{P} =$	17	7	2	-4.25

x_2 (the min for which P is positive) exits the basis. We have discovered a cycle.

Solution

- The solution is made of the possible feasible basis variable, here :

$$\text{IBV}_1 = \{x_4, x_3, x_1, x_7\}\text{IBV}_2 = \{x_4, x_3, x_2, x_7\}$$

as well as all their convex combinations :
- The values of these variables are given by

$$\bar{b}_1 = \{24, 8, 2, 5\}\bar{b}_2 = \{27.2, 11.2, 1.6, 3.4\}$$

respectively, all other non listed variables have value 0.
- Considering only the variables that enter in the cost, the two extreme optima are :

$$e_1 = \begin{bmatrix} x_1 = 2 \\ x_2 = 0 \\ x_3 = 8 \end{bmatrix}, e_2 = \begin{bmatrix} x_1 = 0 \\ x_2 = 1.6 \\ x_3 = 11.2 \end{bmatrix}$$

Solutions

- All the intermediary solutions are given by :

$$\begin{bmatrix} x_1 = 2c \\ x_2 = 1.6 - 1.6c \\ x_3 = 11.2 - 3.2c \end{bmatrix}$$

  with $0 \leq c \leq 1$.
- The cost function is therefore :

$$z = -60x_1 - 30x_2 - 20x_3 = -280$$

  and it is constant for all these solutions.
- For more complex solutions, we can have several variables at zero in P. The set of solution is the convex vectorial space induced by the extremal solutions. To find them we need to realize all the variable substitutions possible. Cycles can be as long as the rank of the matrix, minus one.

Unbounded solutions

- Consider a baker that produces both ordinary and rye breads ;
- Ordinary breads sells for 36 centimes and rye bread for 30 centimes ;
- One ordinary bread requires one unit of raising powder and 60g of flour ; one rye bread requires one unit of raising powder and 50g of flour.
- The bakery has stocks of 5 units of raising powder and 100g of flour.
- We can purchase more raising powder and flour. Raising powder costs 3 centimes per unit, flour cost 4 centimes each 10g.
- Maximize the bakery's profit.

Formulation

- x_1 = number of ordinary breads produced
- x_2 = number of rye breads produced
- x_3 = number of units of raising powder
- x_4 = consumed flour per 10g.
- Revenues = $36x_1 + 30x_2$, costs = $3x_3 + 4x_4$
- Objective = $\max z = 36x_1 + 30x_2 - 3x_3 - 4x_4$
- Constraint 1 : $x_1 + x_2 \leq 5 + x_3$
- Constraint 2 : $6x_1 + 5x_2 \leq 10 + x_4$

Standard form

- We introduce two slack variables for the constraits, x_5, x_6, that are both positive
- the standard form is the following :

$$
\begin{array}{lrrrrrrrrrrrl}
\min\ z = & -36x_1 & - & 30x_2 & + & 3x_3 & + & 4x_4 & & & & & \\
 & x_1 & + & x_2 & - & x_3 & & & + & x_5 & & & = 5 \\
 & 6x_1 & + & 5x_2 & & & - & x_4 & & & + & x_6 & = 10
\end{array}
$$

First iteration

- IBV=$\{x_5, x_6\}$; NBV=$\{x_1, x_2, x_3, x_4\}$
- $\bar{b} = \begin{bmatrix} 5 & 10 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} -36 & -30 & 3 & 4 \end{bmatrix}$ so x_1 is entering.
- $P = \begin{bmatrix} 1 & 6 \end{bmatrix}$
- Ratios = $\begin{bmatrix} 5 & 1.66667 \end{bmatrix}$ so x_6 is exiting.

Second iteration

- IBV=$\{x_5, x_1\}$; NBV=$\{x_2, x_3, x_4, x_6\}$
- $\bar{b} = \begin{bmatrix} 3.333 & 1.667 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} 0 & 3 & -2 & 6 \end{bmatrix}$ so x_4 is entering.
- $P = \begin{bmatrix} 0.167 & -0.167 \end{bmatrix}$
- Ratios = $\begin{bmatrix} 20 & -10 \end{bmatrix}$ so x_5 is exiting.

Third iteration

- IBV=$\{x_4, x_1\}$; NBV=$\{x_2, x_3, x_5, x_6\}$
- $\bar{b} = \begin{bmatrix} 20 & 5 \end{bmatrix}$
- Reduced costs = $\begin{bmatrix} 2 & -9 & 12 & 4 \end{bmatrix}$ so x_3 is entering.
- $P = \begin{bmatrix} -3.333 & -5 \end{bmatrix}$
- Ratios = $\begin{bmatrix} -6 & -1 \end{bmatrix}$ The solution is unbounded.

Solution for the bakery

- With the last IBV, the solution writes :

$$\begin{aligned} 6x_1 \; -x_4 &= 10 \\ x_1 \; -x_3 &= 5 \end{aligned}$$

- Replacing x_1 and x_4 in terms of x_3 by these expressions in the cost yields

$$z = -100 - 9x_3$$

  By augmenting x_3 arbitrarily, z is unbounded towards $-\infty$, while still obeying all the constraints.
- This means the bakery is profitable !